ELEN 4810 Final Exam

Monday, December 18, 2023, 1:10-4:00 PM. Two sheets of handwritten notes are allowed. No electronics of any kind are allowed. Please record your answers in the exam booklet. Raise your hand if you need additional scratch paper.

There are a total of 4 questions. Good luck!

Name:

Uni:

1. Z-transform. A discrete-time LTI system has transfer function

$$H(z) = \frac{(1 + \frac{3}{2}z^{-1})(1 - \frac{3}{2}z^{-1})}{(1 + j\frac{1}{2}z^{-1})(1 - j\frac{1}{2}z^{-1})}. \tag{1}$$

Please answer the following questions:

Part 1. Please plot the pole-zero diagram of $H(z)$, labeling all poles and zeros. You can use the axes on the next page.

Part 2. Which of the following best describes this system?

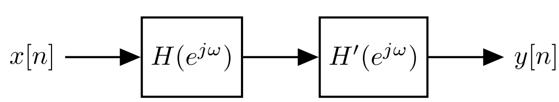
HIGH PASS   LOW PASS   ALL PASS   BAND PASS

Please justify your answer.

Part 3. Could the system be stable and causal? Why or why not?

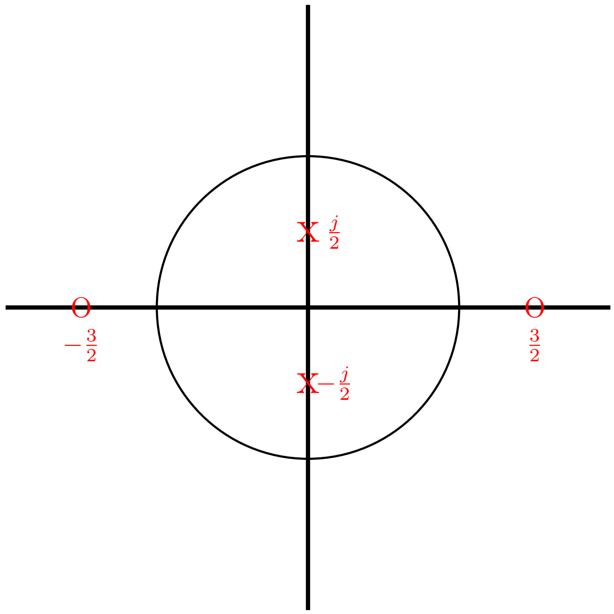
Part 4. Could the impulse response $h[n]$ be real valued? Why or why not?

Part 5. Please determine the transfer function $H'(z)$ of a system which is **causal** and **stable**, *and* ensures for the following system which applies H and H' in series, the output $y[n]$ satisfies $|Y(e^{j\omega})| = |X(e^{j\omega})|$ **for every input** $x[n]$:



Answer to Problem 1: 11 points

Part 1. (2 points)



Part 2. (2 points) BANDPASS. The Fourier transform is maximized at frequencies $\pm\frac{\pi}{2}$, and minimized at frequencies 0 (low frequency) and π (high frequency).

Part 3. (2 points) Yes. All poles are inside the unit circle.

Part 4. (2 points) Yes. Poles and zeros occur in complex conjugate pairs.

Part 5. (3 points) Because $H(z)$ has zeros outside the unit circle, it does not have a causal stable inverse. Using the minimum phase all pass decomposition, we can write

$$H(z) = \underbrace{\frac{(\frac{3}{2}+z^{-1})(1-\frac{3}{2}z^{-1})}{(1+j\frac{1}{2}z^{-1})(1-j\frac{1}{2}z^{-1})}}_{\text{minimum phase}} \times \underbrace{\frac{(1+\frac{3}{2}z^{-1})(1-\frac{3}{2}z^{-1})}{(\frac{3}{2}+z^{-1})(1-\frac{3}{2}z^{-1})}}_{\text{all pass}}.$$

Setting

$$H' = \frac{(1+j\frac{1}{2}z^{-1})(1-j\frac{1}{2}z^{-1})}{(\frac{3}{2}+z^{-1})(1-\frac{3}{2}z^{-1})} \tag{2}$$

we ensure that $|H'(e^{j\omega})H(e^{j\omega})| = 1$ for all ω.

2. Generalized Linear Phase Systems. Consider an FIR generalized linear phase system, with real valued impulse response $h[n]$, transfer function $H(z)$ and zeros $\zeta_1, \ldots, \zeta_{L-1}$.

Please answer the following questions:

Part (i). What are the poles of $H(z)$? For any repeated poles, please indicate their multiplicity. If it is not possible to determine the poles from the given information, please explain why.

Part (ii). Set $h'[n] = (-1)^{n+1} h[n]$. Does the resulting system have generalized linear phase? Why or why not?

Part (iii). Please give an expression for the zeros $\zeta'_1, \ldots, \zeta'_{L-1}$ of $H'(z)$ in terms of $\zeta_1, \ldots, \zeta_{L-1}$.

Part (iv). Suppose we wish for $h'[n]$ to be a **low pass** system. What types of canonical generalized linear phase system/systems should we *not* choose for $h[n]$? Why?

Answer to Problem 2 (12 points):

Part (i) (3 points). This system is FIR, and so all of its poles occur at $z = 0$; there is a pole at $z = 0$ of order $L - 1$.

Part (ii) (3 points). Yes. We have

$$H'(z) \;=\; \sum_{n=0}^{L}(-1)^{n+1}h[n]z^{-n} \;=\; -\sum_{n=0}^{L}h[n](-z)^{-n} \;=\; -H(-z).$$

and so $H'(e^{j\omega}) = -H(e^{j(\omega-\pi)})$ and $\angle H'(e^{j\omega}) = \angle H(e^{j(\omega-\pi)}) + \pi$. If H has generalized linear phase, $\angle H = \alpha\omega + \beta$ for some α, β, and so

$$\angle H'(e^{j\omega}) = \alpha(\omega - \pi) + \beta + \pi = \alpha\omega + (\beta + \pi - \alpha\pi),$$

and so the system has generalized linear phase.

Part (iii) (3 points). Because $H'(z) = -H(-z)$, if ζ_i is a zero of H, $-\zeta_i$ is a zero of H'. Hence, zeros of H' are $-\zeta_1, -\zeta_2, \ldots, -\zeta_{L-1}$.

Part (iv) (3 points). For H' to be low-pass, H needs to be high-pass. We should *not* choose a system with a forced zero at $z = -1$. Hence, we should *not* choose a Type II or Type III GLP system.

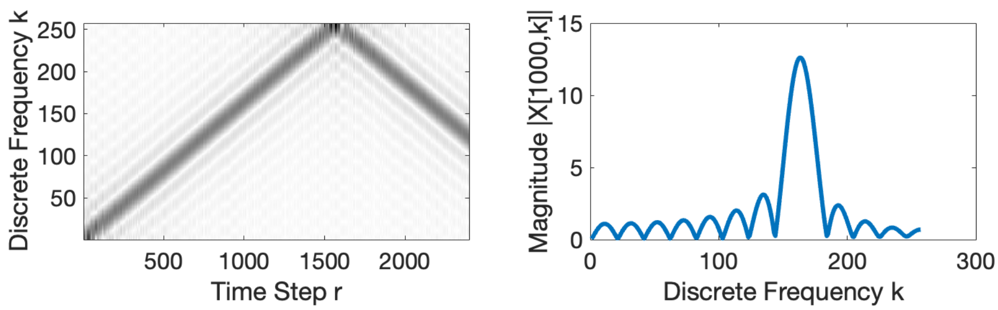
3. Spectrograms. A continuous-time chirp signal

$$x_c(t) = \cos(\alpha t^2)$$

is sampled with a sampling period

$$T_s = 0.03\,\text{seconds}$$

to produce a discrete time signal $x[n]$. We compute the Short-Time Fourier Transform (STFT), $X[r,k]$, using a time stride of $R = 10$ samples, $N = 512$ frequency samples, and a window $w[n]$ of length L. We plot the magnitude of the Short-Time Fourier Transform (STFT) $|X[r,k]|$, for $r = 0, \ldots, 2400$ and $k = 0, \ldots, 255$:



Above, the graph at right plots the vertical slice

$$X[1000,0], X[1000,1], \ldots, X[1000,255].$$

(Note, that $N = 512$; here we only show $X|[r,k]|$ for $k < N/2$).

Please answer the following questions:

Part (a). Please estimate the chirp parameter α. Justify your answer!

Part (b). Why does the spectrogram exhibit a rising line and a falling line?

Part (c). Which of the following windows was used to determine the spectrogram?

RECTANGULAR HAMMING

Please explain your answer!

Part (d). Please estimate the length L of the window, based on the available information.
Note: your estimate does not need to be perfect, but please explain how you arrived at it.

Answer to Problem 3 (11 points):

Part (a) (3 points). We have

$$x[n] = \cos(\alpha T^2 n^2).$$

The instantaneous radian frequency of this cosine is

$$\hat{\omega} = 2\alpha T^2 n.$$

In discrete frequency, this is

$$k = N\omega/2\pi = \alpha N T^2 n/\pi.$$

The graph exhibits frequency

$$k \approx (250/1500)r = r/6 = n/60$$

where we have used that $n = rR$ with $R = 10$. Combining these two equations, we obtain

$$\alpha N T^2 n/\pi = n/60,$$

whence

$$\alpha \approx N T^2 \pi/60 \approx 0.0077.$$

Note: to obtain full credit, students do not need to produce the final numerical value. For full credit, we need the following elements:

- Work with the discrete time signal $x[n]$
- Compute instantaneous frequency correctly, i.e., $\hat{\omega} = 2\alpha T^2 n$ which comes from differentiating $\alpha T^2 n^2$ wrt the n variable
- Convert correctly from ω to k and r to n, using the relationships $\omega = 2\pi k/N$ and $n = rR$.

Part (b) (3 points). The signal $x_c(t)$ can be written as

$$x[n] = \cos(\alpha(nT)^2) = \tfrac{1}{2}e^{j\alpha T^2 n^2} + \tfrac{1}{2}e^{-j\alpha T^2 n^2}.$$

Each of these terms produces a line in the spectrogram. The falling frequency term

$$\tfrac{1}{2}e^{-j\alpha T^2 n^2}$$

has instantaneous frequency $\hat{\omega} = -2\alpha T^2 n$; in the short time fourier transform, this is aliased to radian frequency $2\pi - 2\alpha T^2 n$, which corresponds to discrete frequency

$$k = N - \frac{\alpha T^2 n N}{\pi}.$$

The falling line is caused by this frequency component.

Part (c) (2 points). This is a rectangular window. This can be determined from the large sidelobes, visible as ripples in the magnitude spectrogram.

Part (d) (3 points). The rectangular window has a mainlobe width of $4\pi/L$ in radian frequency ω, where L is the length of the window. From the graph, the mainlobe spans roughly 40 discrete frequency samples.

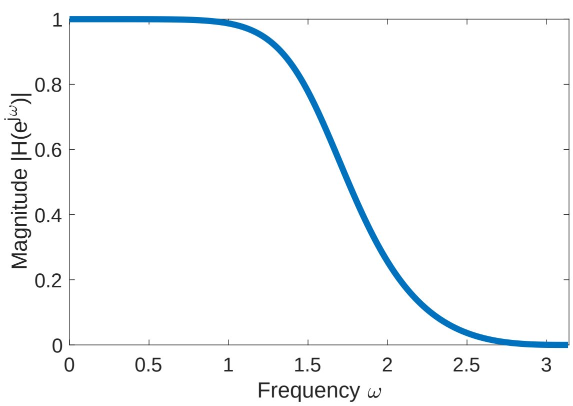
Using the conversion $\omega = 2\pi k/N$, the width of the mainlobe is roughly $80\pi/512$ radians; this gives an estimate of $L = 512 \times 4/80 \approx 25$ samples. (the actual window length is 25 samples).

Note: full credit to answers which use correct process, but do not calculate the final answer (e.g., leave it as $512 \times 4/80$), or which use correct process but have some estimation error

4. IIR Filter Design and Bilinear Transform. We generate a discrete time IIR filter by applying bilinear transformation

$$H(z) = H_c\left(\frac{z-1}{z+1}\right)$$

to a continuous time system, with transfer function $H_c(s)$. Here is the magnitude response $|H(e^{j\omega})|$:



Please answer the following questions:

Part A. Which of the following best describes this filter?

BUTTERWORTH   CHEBYSCHEV I   CHEBYSCHEV II   ELLIPTIC

Part B. Our continuous time filter $H_c(s)$ has poles at

$$s = -1, \quad s = \exp\left(\frac{j2\pi}{3}\right), \quad s = \exp\left(-\frac{j2\pi}{3}\right),$$

and a zero of multiplicity three at $s = \infty$. **What are the poles and zeros of the discrete time system $H(z)$?**

Part C. What is one advantage of FIR filters (e.g., designed by windowing or optimization) compared to the IIR design in part A?

Part D. Suppose we generate a new system, by setting $H'(z) = H(z)H^*(1/z^*)$. **What is the phase response $\angle H'(e^{j\omega})$ of the new system?**

Part E. The system H' has order six (six poles and six zeros). Based on your answer to Part D, please describe one advantage to the system H', compared directly designing an order six system using bilinear transformation.

Part F. Can the system $H'(z)$ be both **stable** and **causal**? Why or why not?

Part G. Please give an expression for the impulse response $h'[n]$ in terms of the impulse response $h[n]$.

Answer to Problem 4 (14 points):

Part A. (2 points) Butterworth – filter is monotone in passband and stopband.

Part B. (2 points) Using the bilinear transform relationship $z = \frac{1+s}{1-s}$, we find that the new system has a zero of order 3 at $z = -1$. The poles are

$$\begin{aligned}
\rho_1 &= \frac{1 + -1}{1-(-1)} = 0 \\
\rho_2 &= \frac{1 + (-\frac{1}{2} + j\sqrt{3}/2)}{1-(-\frac{1}{2} + j\sqrt{3}/2)} = \frac{1 + j\sqrt{3}}{3 - j\sqrt{3}} \\
\rho_3 &= \frac{1 + (-\frac{1}{2} - j\sqrt{3}/2)}{1-(-\frac{1}{2} - j\sqrt{3}/2)} = \frac{1 - j\sqrt{3}}{3 + j\sqrt{3}}.
\end{aligned}$$

Part C. (2 points) for FIR designs, we can enforce generalized linear phase.

Part D. (2 points) We have

$$H'(e^{j\omega}) = H(e^{j\omega})H^*(1/e^{-j\omega}) = H(e^{j\omega})H^*(e^{j\omega}) = |H(e^{j\omega})|^2.$$

The phase response is

$$\angle H'(e^{j\omega}) = 0$$

for all ω.

Part E. (2 points) The system in part D has generalized linear phase (actually, zero phase). An order six iir system designed by BLT will not have generalized linear phase.

Part F. (2 points) No. The poles of H' are ρ_1, ρ_2, ρ_3, *and* $1/\rho_1^*$, $1/\rho_2^*$, $1/\rho_3^*$. The latter three poles lie outside the unit circle.

Part G. (2 points) We have $H'(z) = H(z)H^*(1/z^*)$. Write

$$\tilde{h}[n] = h^*[-n].$$

Using the Z-transform relationship

$$h^*[-n] \to H^*(1/z^*)$$

we have

$$h'[n] = (h * \tilde{h})[n].$$